# MATH 303 – Measures and Integration
# Final Exam Solutions

**Problem 1.** Give a full statement of the following theorems related to integration of functions:

- Monotone convergence theorem
- Fatou's lemma

Prove that the monotone convergence theorem and Fatou's lemma are equivalent. That is, give a proof of the following two implications:

- (monotone convergence theorem) $\implies$ (Fatou's lemma)
- (Fatou's lemma) $\implies$ (monotone convergence theorem)

**Solution:** The theorem statements are given in the lecture notes (Theorems 3.10 and 3.13).

**MCT $\implies$ Fatou:** See the proof of Fatou's lemma (Theorem 3.13) in the lecture notes.

**Fatou $\implies$ MCT:** Let $0 \leq f_1 \leq f_2 \leq \dots$ be an increasing sequence of nonnegative measurable functions defined on a measure space $(X, \mathcal{B}, \mu)$. Let $f = \lim_{n\to\infty} f_n$. We want to show

$$\int_X f \, d\mu = \lim_{n\to\infty} \int_X f_n \, d\mu. \tag{1}$$

By monotonicity of the integral,

$$\int_X f \, d\mu \geq \int_X f_n \, d\mu$$

for each $n \in \mathbb{N}$, so

$$\int_X f \, d\mu \geq \limsup_{n\to\infty} \int_X f_n \, d\mu. \tag{2}$$

On the other hand, by Fatou's lemma,

$$\int_X f \, d\mu \leq \liminf_{n\to\infty} \int_X f_n \, d\mu. \tag{3}$$

The inequalities (2) and (3) combined establish the desired identity (1).

**Problem 2.** In this course, we defined an outer measure to be a function $\mu^* : \mathscr{P}(X) \to [0, \infty]$ such that $\mu^*(\emptyset) = 0$ and $\mu^*$ is monotone and countably subadditive.

(a) What does it mean for $\mu^*$ to be monotone?

(b) What does it mean for $\mu^*$ to be countably subadditive?

(c) Show that a function $\mu^* : \mathscr{P}(X) \to [0, \infty]$ is an outer measure if and only if $\mu^*(\emptyset) = 0$ and $\mu^*$ satisfies the following property: if $A \subseteq X$, $(B_n)_{n\in\mathbb{N}}$ is a sequence of subsets of $X$, and $A \subseteq \bigcup_{n\in\mathbb{N}} B_n$, then $\mu^*(A) \leq \sum_{n=1}^{\infty} \mu^*(B_n)$.

**Solution:** (a) Monotone: if $E \subseteq F$, then $\mu^*(E) \leq \mu^*(F)$.

(b) Countably subbaditive: if $(E_n)_{n\in\mathbb{N}}$ is a sequence of subsets of $X$, then $\mu^*\left(\bigcup_{n\in\mathbb{N}} E_n\right) \leq \sum_{n=1}^{\infty} \mu^*(E_n)$.

(c) Suppose $\mu^*$ is an outer measure. Let $A \subseteq X$, and suppose $(B_n)_{n\in\mathbb{N}}$ is a sequence of subsets of $X$ such that $A \subseteq \bigcup_{n\in\mathbb{N}} B_n$. Applying monotonicity, $\mu^*(A) \leq \mu^*\left(\bigcup_{n\in\mathbb{N}} B_n\right)$. Then applying countable subadditivity, $\mu^*\left(\bigcup_{n\in\mathbb{N}} B_n\right) \leq \sum_{n=1}^{\infty} \mu^*(B_n)$. Combining these two steps, we conclude $\mu^*(A) \leq \sum_{n=1}^{\infty} \mu^*(B_n)$.

Conversely, suppose $\mu^*$ satisfies $\mu^*(\emptyset) = 0$ and if $A \subseteq X$, $(B_n)_{n\in\mathbb{N}}$ is a sequence of subsets of $X$, and $A \subseteq \bigcup_{n\in\mathbb{N}} B_n$, then $\mu^*(A) \leq \sum_{n=1}^{\infty} \mu^*(B_n)$. We must check that $\mu^*$ is monotone and countably subadditive.

- MONOTONE: Suppose $E \subseteq F$. Let $B_1 = F$ and $B_n = \emptyset$ for $n \geq 2$. Then $E \subseteq \bigcup_{n\in\mathbb{N}} B_n$, so $\mu^*(E) \leq \sum_{n=1}^{\infty} \mu^*(B_n) = \mu^*(F)$.
- COUNTABLY SUBADDITIVE: Suppose $(E_n)_{n\in\mathbb{N}}$ is a sequence of subsets of $X$. Then $\bigcup_{n\in\mathbb{N}} E_n \subseteq \bigcup_{n\in\mathbb{N}} E_n$, so $\mu^*\left(\bigcup_{n\in\mathbb{N}} E_n\right) \leq \sum_{n=1}^{\infty} \mu^*(E_n)$.

**Problem 3.** Let $X$ be an uncountable set.

(a) Prove that the collection $\mathcal{B} = \{E \subseteq X : E \text{ is countable or } X \setminus E \text{ is countable}\}$ is a $\sigma$-algebra on $X$.

(b) Define a function $\mu : \mathcal{B} \to \{0, 1\}$ by $\mu(E) = 0$ if $E$ is countable and $\mu(E) = 1$ if $X \setminus E$ is countable. Prove that $\mu$ is a measure.

(c) Describe the collection of measurable functions from $X$ to $\mathbb{R}$ and compute their integrals with respect to $\mu$.

**Solution:** In the solution below, we say that a set $E$ is *co-countable* if its complement $X \setminus E$ is countable.

(a) Let us check each of the axioms of a $\sigma$-algebra.

- $X \setminus X = \emptyset$ is countable, so $X \in \mathcal{B}$.
- The condition "$E$ is countable or $X \setminus E$ is countable" is symmetric in a set $E$ and its complement, so $\mathcal{B}$ is closed under complements.
- Let $(E_n)_{n\in\mathbb{N}}$ be a sequence of elements of $\mathcal{B}$. If each of the sets $E_n$ is countable, then their union $\bigcup_{n\in\mathbb{N}} E_n$ is also countable. On the other hand, if $X \setminus E_{n_0}$ is countable for some $n_0 \in \mathbb{N}$, then $X \setminus \bigcup_{n\in\mathbb{N}} E_n \subseteq X \setminus E_{n_0}$ is countable. In either case, $\bigcup_{n\in\mathbb{N}} E_n \in \mathcal{B}$.

(b) The empty set is countable, so $\mu(\emptyset) = 0$. Suppose $(E_n)_{n\in\mathbb{N}}$ is a sequence of pairwise disjoint measurable sets, and let $E = \bigsqcup_{n\in\mathbb{N}} E_n$. We want to show $\mu(E) = \sum_{n=1}^{\infty} \mu(E_n)$. We split into two cases.

<u>Case 1</u>: $E_n$ is countable for every $n \in \mathbb{N}$.
Then $E$ is also countable, so $\mu(E) = 0$ and $\sum_{n=1}^{\infty} \mu(E_n) = \sum_{n=1}^{\infty} 0 = 0$.

Case 2: $X \setminus E_{n_0}$ is countable for some $n_0 \in \mathbb{N}$.
Since the sets are disjoint, if $n \neq n_0$, then $E_n \subseteq X \setminus E_{n_0}$ is countable. Therefore,

$$\mu(E_n) = \begin{cases} 1, & \text{if } n = n_0; \\ 0, & \text{if } n \neq n_0. \end{cases}$$

Hence, $\sum_{n=1}^{\infty} \mu(E_n) = 1$. Moreover, $X \setminus E \subseteq X \setminus E_{n_0}$ is countable, so $\mu(E) = 1$.

(c) Claim 1: A function $f : X \to \mathbb{R}$ is measurable if and only if there exists $c \in \mathbb{R}$ such that $\{f = c\}$ is co-countable.

*Proof of Claim 1.* First, suppose $\{f = c\}$ is co-countable. Let $E = \{f = c\}$, and let $S = X \setminus E$. Then we can express $f$ as a countable sum by writing $f = c\mathbb{1}_E + \sum_{x \in S} f(x)\mathbb{1}_{\{x\}}$. A countable sum of measurable functions is measurable, and scalar multiples of measurable functions are measurable, so it suffices to check that each of the functions $\mathbb{1}_E$ and $\mathbb{1}_{\{x\}}$ is measurable. But $E \in \mathcal{B}$ by assumption, and each of the sets $\{x\}$ is countable so belongs to $\mathcal{B}$. Indicator functions of measurable sets are measurable, so we conclude that $f$ is a measurable function.

Conversely, suppose $f$ is measurable. Then $\{f > t\} \in \mathcal{B}$ for every $t \in \mathbb{R}$. By continuity of $\mu$ from below, $\lim_{n \to -\infty} \mu(\{f > n\}) = 1$. Similarly, by continuity from above (which applies since $\mu$ is a finite measure), we have $\lim_{n \to \infty} \mu(\{f > n\}) = \mu(\emptyset) = 0$. Since $\mu$ only takes values 0 and 1, this means that $\{f > t\}$ is countable for all sufficiently large $t \in \mathbb{R}$ and co-countable for all sufficiently small $t \in \mathbb{R}$. Let $c = \sup\{t \in \mathbb{R} : \{f > t\} \text{ is co-countable}\}$. Since $\{f \geq c\} = \bigcap_{n \in \mathbb{N}} \{f > c - \frac{1}{n}\}$, we have $\mu(\{f \geq c\}) = 1$ by continuity from above. Similarly, writing $\{f > c\} = \bigcup_{n \in \mathbb{N}} \{f > c + \frac{1}{n}\}$ and applying continuity from below, $\mu(\{f > c\}) = 0$. Hence, $\{f = c\} = \{f \geq c\} \setminus \{f > c\}$ is co-countable. □

Claim 2: Given a measurable function $f : X \to \mathbb{R}$, let $c \in \mathbb{R}$ such that $\{f = c\}$ is co-countable by claim 1. Then $\int_X f \, d\mu = c$.

*Proof of Claim 2.* Another way of saying $\{f = c\}$ is co-countable is to say that $f = c$ $\mu$-a.e. Therefore, $f$ and $c$ have the same integral, so $\int_X f \, d\mu = \int_X c \, d\mu = c\mu(X) = c$. □

**Problem 4.** Let $(X, \mathcal{B}, \mu)$ be a measure space, and let $f : X \to \mathbb{C}$ be a measurable function. Suppose $f$ is integrable (with respect to $\mu$). Prove that for any $\varepsilon > 0$, there exists $M > 0$ such that

$$\int_{\{|f| > M\}} |f| \, d\mu < \varepsilon.$$

**Solution: Method 1: Defining a measure.** Define $\nu : \mathcal{B} \to [0, \infty)$ by $\nu(E) = \int_E |f| \, d\mu$. Note that $\nu$ is a finite measure on $(X, \mathcal{B})$, since $f$ is integrable. For each $n \in \mathbb{N}$, let $E_n = \{|f| > n\}$. Note that $E_1 \supseteq E_2 \supseteq \dots$ and $\bigcap_{n \in \mathbb{N}} E_n = \emptyset$. Then by continuity of the measure $\nu$ from above (and finiteness of $\nu$), $\lim_{n \to \infty} \nu(E_n) = \nu(\emptyset) = 0$. Thus, given $\varepsilon > 0$, there exists $M \in \mathbb{N}$ such that $n \geq M \implies \nu(E_n) < \varepsilon$. This value of $M$ has the desired property.

**Method 2: Dominated convergence.** Let $E_n = \{|f| > n\}$, and let $g_n = |f|\mathbb{1}_{E_n}$. Note that $\bigcap_{n \in \mathbb{N}} E_n = \emptyset$, so $\mathbb{1}_{E_n}(x) \to 0$ as $n \to \infty$ for each $x \in X$. Therefore, $g_n \to 0$ pointwise.

Moreover, $|g_n| \leq |f| \in L^1(\mu)$ for every $n \in \mathbb{N}$. Thus, by the dominated convergence theorem, $\int_X g_n \, d\mu \to 0$. Given $\varepsilon > 0$, we may therefore find $M \in \mathbb{N}$ such that

$$\int_{\{|f|>M\}} |f| \, d\mu = \int_X g_M \, d\mu < \varepsilon.$$

**Method 3: Monotone convergence.** Define a sequence of functions $g_n : X \to [0, \infty]$ by $g_n = |f| \cdot \mathbb{1}_{\{|f| \leq n\}}$ so that $0 \leq g_1 \leq g_2 \leq \dots$ and $\lim_{n\to\infty} g_n = f$. By the monotone convergence theorem,

$$\lim_{n\to\infty} \int_X g_n \, d\mu = \int_X |f| \, d\mu.$$

Since $f$ is integrable, the integral on the right hand size is finite. Thus, given $\varepsilon > 0$, we may find $M \in \mathbb{N}$ such that

$$\int_X |f| \, d\mu - \int_X g_M \, d\mu < \varepsilon.$$

But from the definition of $g_M$ and linearity of the integral, we have

$$\int_{\{|f|>M\}} |f| \, d\mu = \int_{X\setminus\{|f|\leq M\}} |f| \, d\mu = \int_X |f| \, d\mu - \int_{\{|f|\leq M\}} |f| \, d\mu = \int_X |f| \, d\mu - \int_X g_M \, d\mu,$$

so we have found the desired value of $M$.

**Problem 5.** Let $(X, \mathcal{B})$ be a measurable space, let $\mu : \mathcal{B} \to [0, \infty]$ be a measure, and let $\nu : \mathcal{B} \to [0, \infty)$ be a finite measure. Prove that the following are equivalent:

(i) for any $E \in \mathcal{B}$, if $\mu(E) = 0$, then $\nu(E) = 0$;

(ii) for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $E \in \mathcal{B}$ and $\mu(E) < \delta$, then $\nu(E) < \varepsilon$.

Show that the implication (i) $\implies$ (ii) may fail if $\nu$ is an infinite measure.

**Solution:** (i) $\implies$ (ii).

**Method 1: Borel–Cantelli.** Suppose (i) holds, and suppose for contradiction that (ii) fails. Then there exists $\varepsilon > 0$ and sets $E_1, E_2, \dots \in \mathcal{B}$ with $\mu(E_n) < 2^{-n}$ such that $\nu(E_n) \geq \varepsilon$ for every $n \in \mathbb{N}$. Let $E = \bigcap_{k\in\mathbb{N}} \bigcup_{n\geq k} E_n$. By the Borel–Cantelli lemma, $\mu(E) = 0$, so by (i), we have $\nu(E) = 0$. However, since $\nu$ is finite, we can apply continuity from above to deduce

$$\nu(E) = \lim_{k\to\infty} \nu\left(\bigcup_{n\geq k} E_n\right) \geq \limsup_{k\to\infty} \nu(E_k) \geq \varepsilon.$$

This is a contradiction.

**Method 2: Radon–Nikodym.** Suppose (i) holds. We will first reduce to the case that $\mu$ is $\sigma$-finite. Note that (ii) is equivalent to the statement: if $(E_n)_{n\in\mathbb{N}}$ is a sequence of measurable sets and $\lim_{n\to\infty} \mu(E_n) = 0$, then $\lim_{n\to\infty} \nu(E_n) = 0$. Given a sequence of measurable sets $(E_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} \mu(E_n) = 0$, we may assume that $\mu(E_n) < \infty$ for all $n \in \mathbb{N}$ by disregarding finitely many elements of the sequence if necessary. Then the set $X_0 = \bigcup_{n\in\mathbb{N}} E_n$ is a $\sigma$-finite set for $\mu$, so by restricting the problem to $X_0 \subseteq X$, we may assume without loss of generality that $\mu$ is a $\sigma$-finite measure.

Now by the Radon–Nikodym theorem, let $f : X \to [0, \infty]$ be a measurable function such that $d\nu = f\ d\mu$. Since $\nu$ is a finite measure, $f \in L^1(\mu)$. Let $\varepsilon > 0$. By the definition of the integral, there exists a simple function $0 \le s \le f$ such that $\int_X s\ d\mu > \int_X f\ d\mu - \frac{\varepsilon}{2}$. Let $M = \max_{x \in X} |s(x)|$ and put $\delta = \frac{\varepsilon}{2 \max_{x \in X} |s(x)|}$. If $E \in \mathcal{B}$ and $\mu(E) < \delta$, then

$$\nu(E) = \int_E f\ d\mu = \underbrace{\int_E s\ d\mu}_{\le M\mu(E)} + \underbrace{\int_E (f - s)\ d\mu}_{< \frac{\varepsilon}{2}} < \varepsilon.$$

(ii) $\implies$ (i). Conversely, assume (ii) holds, and let $E \in \mathcal{B}$ with $\mu(E) = 0$. Then for any $\varepsilon > 0$, let $\delta$ be given as in (ii). Since $\mu(E) < \delta$, we conclude $\nu(E) < \varepsilon$. But $\varepsilon$ was arbitrary, so $\nu(E) = 0$.

For a counterexample, one may take $\mu$ to be any measure for which there exists sets of arbitrarily small positive $\mu$-measure (for example, the Lebesgue measure on $[0, 1]$) and define $\nu = \infty \cdot \mu$. Then (i) is satisfied but (ii) fails.

**Problem 6.** Consider the set of integers $\mathbb{Z}$ as a discrete topological space.

(a) Describe the space $C_c(\mathbb{Z})$ of compactly supported continuous functions on $\mathbb{Z}$.

(b) Describe the positive linear functionals on $C_c(\mathbb{Z})$.

(c) Let $\varphi : C_c(\mathbb{Z}) \to \mathbb{C}$ be a positive linear functional, and let $\mu$ be the Radon measure representing $\varphi$ via the Riesz representation theorem. When is $\mu$ a finite measure? (Give a characterization in terms of properties of $\varphi$.)

**Solution:** (a) Every function on a discrete space is continuous. Moreover, a subset $K \subseteq \mathbb{Z}$ is compact if and only if $K$ is finite. Therefore, $C_c(\mathbb{Z})$ is the family of functions $f : \mathbb{Z} \to \mathbb{C}$ that vanish outside of a finite set. This can be identified with the direct sum $\bigoplus_{n \in \mathbb{Z}} \mathbb{C}$ by taking as a basis the functions $e_n = \mathbb{1}_{\{n\}}$.

(b) A linear functional is determined by its values on a basis. Given a positive linear functional $\varphi : C_c(\mathbb{Z}) \to \mathbb{C}$, define $a_\varphi : \mathbb{Z} \to [0, \infty)$ by $a_\varphi(n) = \varphi(e_n)$. Note that $a_\varphi(n) \ge 0$, since $e_n \ge 0$ and $\varphi$ is positive.

On the other hand, given an arbitrary nonnegative function $a : \mathbb{Z} \to [0, \infty)$, we may define a positive linear functional $\varphi_a : C_c(\mathbb{Z}) \to \mathbb{C}$ by $\varphi_a(f) = \sum_{n \in \mathbb{Z}} f(n)a(n)$. (This sum has only finitely many nonzero terms so is well-defined.) Linearity of the functional is clear from the definition, as is positivity, since a sum of nonnegative numbers will be nonnegative.

We have therefore identified the space of positive linear functionals with the space of nonnegative functions $a : \mathbb{Z} \to [0, \infty)$.

(c) From the description in (b), let $a : \mathbb{Z} \to [0, \infty)$ such that $\varphi(f) = \sum_{n \in \mathbb{Z}} f(n)a(n)$. If $E \subseteq \mathbb{Z}$ is a finite set, then $\mathbb{1}_E \in C_c(X)$, so $\mu(E) = \varphi(\mathbb{1}_E) = \sum_{n \in E} a(n)$. Therefore, by continuity from below of the measure $\mu$, we have $\mu(\mathbb{Z}) = \lim_{N \to \infty} \mu(\mathbb{Z} \cap [-N, N]) = \lim_{N \to \infty} \sum_{n=-N}^{N} a(n) = \sum_{n=-\infty}^{\infty} a(n)$. It follows that $\mu$ is a finite measure if and only if

$\sum_{n=-\infty}^{\infty} a(n) < \infty$. In other words, $\mu$ is finite if and only if $a \in \ell^1(\mathbb{Z})$.

**Problem 7.** Let $X$ be an LCH space and $\mu : \mathrm{Borel}(X) \to [0, \infty]$ a Radon measure on $X$. Prove that if $K \subseteq X$ is compact, then

$$\mu(K) = \inf \left\{ \int_X f \, d\mu : f \in C_c(X), K \prec f \right\}.$$

**Solution:** Let $K \subseteq X$ be compact. If $f \in C_c(X)$ and $K \prec f$, then $\mathbb{1}_K \leq f$, so $\mu(K) = \int_X \mathbb{1}_K \, d\mu \leq \int_X f \, d\mu$ by monotonicity of the integral.

Let us prove the other inequality. Note that $\mu(K) < \infty$, since $\mu$ is a locally finite measure. Let $\varepsilon > 0$. Since $\mu$ is outer regular, there exists an open set $V \subseteq X$ such that $K \subseteq V$ and $\mu(V) < \mu(K) + \varepsilon$. By Urysohn's lemma, let $f \in C_c(X)$ with $K \prec f \prec V$. Then $\int_X f \, d\mu \leq \int_X \mathbb{1}_V \, d\mu = \mu(V) < \mu(K) + \varepsilon$. But $\varepsilon > 0$ was arbitrary, so this proves

$$\inf \left\{ \int_X f \, d\mu : f \in C_c(X), K \prec f \right\} \leq \mu(K).$$

**Problem 8.** Let $F, G : \mathbb{R} \to \mathbb{R}$ be increasing, right-continuous functions with $F(0) = G(0) = 0$. Let $\mu_F$ and $\mu_G$ be the Lebesgue–Stieltjes measures with distribution functions $F$ and $G$ respectively. Show that if either $F$ is continuous or $G$ is continuous, then

$$\int_{(a,b]} F \, d\mu_G + \int_{(a,b]} G \, d\mu_F = F(b)G(b) - F(a)G(a).$$

**Solution:** Both the left hand side and the right hand side are symmetric in $F$ and $G$, so we will assume that $F$ is continuous.

Consider the region $R = \{(x, y) \in \mathbb{R}^2 : a < x < y \leq b\}$. Let $\mu_F \otimes \mu_G$ denote the product measure of $\mu_F$ and $\mu_G$. (This measure is uniquely determined, since Lebesgue–Stieltjes measures are $\sigma$-finite.) By Tonelli's theorem, we may compute $(\mu_F \otimes \mu_G)(R)$ as an iterated integral in two ways:

$$(\mu_F \otimes \mu_G)(R) = \int_{\mathbb{R}^2} \mathbb{1}_R \, d(\mu_F \otimes \mu_G) = \int_{(a,b]} \underbrace{\mu_F((a, y))}_{F(y)-F(a)} \, d\mu_G(y) = \int_{(a,b]} F \, d\mu_G - F(a) \left(G(b) - G(a)\right).$$

and

$$(\mu_F \otimes \mu_G)(R) = \int_{\mathbb{R}^2} \mathbb{1}_R \, d(\mu_F \otimes \mu_G) = \int_{(a,b]} \underbrace{\mu_G((x, b])}_{G(b)-G(x)} \, d\mu_F(x) = G(b) \left(F(b) - F(a)\right) - \int_{(a,b]} G \, d\mu_F.$$

Combining these two computations, we have

$$\int_{(a,b]} F \, d\mu_G + \int_{(a,b]} G \, d\mu_F = G(b) \left(F(b) - F(a)\right) + F(a) \left(G(b) - G(a)\right) = F(b)G(b) - F(a)G(a)$$

as desired.